Cutoff Phenomena for Guided Waves in Moving Media

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Abstract—This paper treats the propagation of electromagnetic waves in the interior of a waveguide that is filled with a moving medium. The medium is assumed to be homogeneous, isotropic, and lossless, and to move with a constant velocity along the axis of the waveguide. The Maxwell-Minkowski equations for the electromagnetic fields are solved by means of a pair of vector potential functions similar to those frequently used for stationary media. The fields inside the waveguide are derived for both rectangular and cylindrical waveguides.

The well-known cutoff phenomenon for a waveguide is found to be modified in an interesting way when the medium inside the waveguide is moving The results show that for a slowly moving medium (a medium for which $n\beta<1$, where n is the index of refraction and β is the velocity of the medium divided by the velocity of light in vacuum), there are two critical frequencies, separating three frequency ranges in each of which there is a different type of propagation. For a high-speed medium ($n\beta>1$), it is found that there is no cutoff phenomenon at all, although there is one critical frequency separating two frequency ranges in which the propagation is different.

Introduction

THIS PAPER considers the propagation of electromagnetic waves in the interior of a waveguide that is filled with a moving medium. The medium is assumed to be homogeneous, isotropic, and lossless, with constitutive parameters μ and ϵ, and to move at constant velocity $\bar{v}$ along the axis of the waveguide. The waveguide is assumed to have perfectly conducting walls and to be infinitely long. The same problem has been discussed by Collier and Tai [1] under the assumption that the velocity of the medium is much smaller than that of light. In this paper we shall treat the case where the velocity of the medium can have any value up to the velocity of light. The purpose of the paper is to show how the familiar "cutoff" phenomenon for a waveguide is modified when the medium inside is moving. This effect is not apparent when the velocity of the medium is assumed to be small.

Development of the Theory

The electromagnetic fields inside a waveguide are governed by Maxwell's equations

$$\nabla \times E = -\frac{\partial \bar{B}}{\partial t} \tag{1}$$

$$\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t} + \bar{J} \tag{2}$$

$$\nabla \cdot \bar{D} = \rho \tag{3}$$

$$\nabla \cdot \bar{B} = 0 \tag{4}$$

which, as we know from the Special Theory of Relativity, are valid for any medium, moving or stationary. In (1)–(4), $\bar{E}$ and $\bar{H}$ are the electric and magnetic field intensities, $\bar{D}$ and $\bar{B}$ are the electric and magnetic flux densities, and ρ and $\bar{J}$ are the charge and current densities. All quantities are measured in MKS units in a coordinate system that is stationary with respect to the walls of the waveguide.

Because the medium is moving, the constitutive relations are different from those that would apply for a stationary medium. The modified constitutive relations for a medium moving with constant velocity were first derived correctly by Minkowski [2], and his results have been put into a compact form by Tai [3]. Assuming the velocity of the medium is in the z-direction, $\bar{v}=v\hat{z}$, the result is

$$\bar{D} = \epsilon\bar{\bar{\alpha}}\cdot\bar{E} + \bar{\Omega}\times\bar{H} \tag{5}$$

$$\bar{B} = \mu\bar{\bar{\alpha}}\cdot\bar{H} - \bar{\Omega}\times\bar{E} \tag{6}$$

where

$$\bar{\Omega} = \frac{(n^2-1)\beta}{(1-n^2\beta^2)c}\hat{z} \tag{7}$$

$$\bar{\bar{\alpha}} = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

$$a = \frac{1-\beta^2}{1-n^2\beta^2} \tag{9}$$

$$n = \sqrt{\mu\epsilon/\mu_0\epsilon_0} \tag{10}$$

$$\beta = v/c \tag{11}$$

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}} = \text{the velocity of light in vacuum} \tag{12}$$

and where μ and ϵ are the permeability and permittivity of the medium.[1] Equations (5) and (6) are valid for any velocity of the medium up to the speed of light.

In this discussion we shall be interested in the normal modes associated with the interior of the waveguide. We assume that in the region under consideration there are

[1] μ and ϵ are measured in a reference frame that is stationary with respect to the medium.

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no sources, and all fields are harmonic with time-dependence $e^{-i\omega t}$. Since the medium is also assumed to be lossless, we then have $\bar{J}=0$ and $\rho=0$. On substituting (5) and (6) into (1)–(4), the Maxwell equations become

$$\bar{D}_1 \times \bar{E} = i\omega\mu\bar{\bar{\alpha}}\cdot\bar{H} \tag{13}$$

$$\bar{D}_1 \times \bar{H} = -i\omega\epsilon\bar{\bar{\alpha}}\cdot\bar{E} \tag{14}$$

$$\bar{D}_1\cdot(\epsilon\bar{\bar{\alpha}}\cdot\bar{E}) = 0 \tag{15}$$

$$\bar{D}_1\cdot(\mu\bar{\bar{\alpha}}\cdot\bar{H}) = 0 \tag{16}$$

where $\bar{D}_1$ is the differential operator

$$\bar{D}_1 = \nabla + i\omega\bar{\Omega}. \tag{17}$$

Two types of potential functions will be introduced to describe the electromagnetic fields in the waveguide.[2] Since $\bar{D}_1\cdot\bar{D}_1\times\bar{W}=0$ and $\bar{D}_1\times\bar{D}_1 U=0$ for any vector $\bar{W}$ and scalar U whose components have continuous second partial derivatives, we may write

$$\mu\bar{\bar{\alpha}}\cdot\bar{H}^e = \bar{D}_1 \times A \tag{18}$$

where $\bar{A}$ is a suitable vector potential function, and where the superscript "e" indicates that the fields associated with $\bar{A}$ are "electric" (TM) modes. Substituting (18) into (13) gives

$$\bar{D}_1 \times \ \bar{E}^e - i\omega A] = 0, \tag{19}$$

and hence we may set

$$\bar{E}^e = i\omega A - \bar{D}_1 U \tag{20}$$

where U is a suitable scalar potential function.

If we define another vector function $\bar{A}_1$ such that

$$A_1 = \bar{\bar{\alpha}}\cdot A \tag{21}$$

and impose the gauge relation

$$\bar{D}_1\cdot A_1 = i\omega\mu\epsilon a^2 U \tag{22}$$

between $\bar{A}_1$ and U, it is not difficult to show in terms of cartesian coordinates that $\bar{A}_1$ has to satisfy the equation

$$(\bar{D}_a\cdot\bar{D}_1)\bar{A}_1 + k^2 a\bar{A}_1 = 0 \tag{23}$$

where

$$\bar{D}_a = \nabla_a + \frac{i\omega}{a}\bar{\Omega} \tag{24}$$

$$\nabla_a = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{1}{a}\frac{\partial}{\partial z} \tag{25}$$

$$k = \omega\sqrt{\mu\epsilon}. \tag{26}$$

[2] The derivation of the potential functions given here closely follows that given previously by Collier and Tai [1], and by Tai [4].

Equation (23), when written out, reads

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{a}\frac{\partial^2}{\partial z^2} + \frac{2i\omega\Omega}{a}\frac{\partial}{\partial z} - \frac{\omega^2\Omega^2}{a} + k^2 a\right]\bar{A}_1 = 0 \tag{27}$$

The field vectors are then given in terms of $\bar{A}_1$ as

$$\bar{E}^e = i\omega\bar{\bar{\alpha}}^{-1}\cdot\bar{A}_1 - \frac{\bar{D}_1(\bar{D}_1\cdot\bar{A}_1)}{i\omega\mu\epsilon a^2} \tag{28}$$

$$\bar{H}^e = \frac{1}{\mu}\bar{\bar{\alpha}}^{-1}\cdot[\bar{D}_1 \times (\bar{\bar{\alpha}}^{-1}\cdot\bar{A}_1)] \tag{29}$$

where $\bar{\bar{\alpha}}^{-1}$ is the inverse of $\bar{\bar{\alpha}}$.

A similar procedure can be followed to find the equations satisfied by the potential functions $\bar{F}$ and V associated with fields $\bar{E}^m$ and $\bar{H}^m$ of magnetic type. The result is given as follows:

$$\bar{E}^m = -\frac{1}{\epsilon}\bar{\bar{\alpha}}^{-1}\cdot[\bar{D}_1 \times (\bar{\bar{\alpha}}^{-1}\cdot\bar{F}_1)] \tag{30}$$

$$\bar{H}^m = i\omega\bar{\bar{\alpha}}^{-1}\cdot\bar{F}_1 - \frac{\bar{D}_1(\bar{D}_1\cdot\bar{F}_1)}{i\omega\mu\epsilon a^2} \tag{31}$$

where $\bar{F}$ and $\bar{F}_1$ are related by

$$\bar{F}_1 = \bar{\bar{\alpha}}\cdot\bar{F}. \tag{32}$$

The gauge condition imposed on $\bar{F}_1$ and V is

$$\bar{D}_1\cdot\bar{F}_1 = i\omega\mu\epsilon a^2 V. \tag{33}$$

$\bar{F}_1$ has to satisfy the same equation as $\bar{A}_1$.

The field solution in the waveguide can be divided into two basic modes, TE and TM. We assume that the z-axis is the longitudinal axis of the waveguide. For TM modes, the field components may be derived from an electric vector potential function $\bar{A}=\hat{z}\bar{A}$. For TE modes, the field components may be derived from a magnetic vector potential $\bar{F}=\hat{z}\bar{F}$. For this particular case where $\bar{A}$ and $\bar{F}$ have only one component in the z-direction, we have

$$\bar{A}_1 = \hat{z}A_1 = \bar{\bar{\alpha}}\cdot\bar{A} = A = \hat{z}A \tag{34}$$

$$\bar{F}_1 = \hat{z}F_1 = \bar{\bar{\alpha}}\cdot\bar{F} = \bar{F} = \hat{z}F. \tag{35}$$

The Rectangular Waveguide

For the rectangular waveguide, we consider the geometry shown in Fig. 1.

The vector potential functions satisfying the appropriate boundary conditions for this geometry are

$$A = \hat{z}A = \hat{z}A_0 \sin\frac{m\pi}{x_0}x \sin\frac{l\pi}{y_0}y e^{ihz} \tag{36}$$

and

$$\bar{F} = \hat{z}\bar{F} = \hat{z}F_0 \cos\frac{m\pi}{x_0}x \cos\frac{l\pi}{y_0}y e^{ihz}. \tag{37}$$

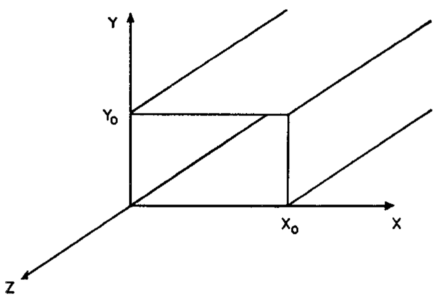

Fig. 1. The rectangular waveguide.

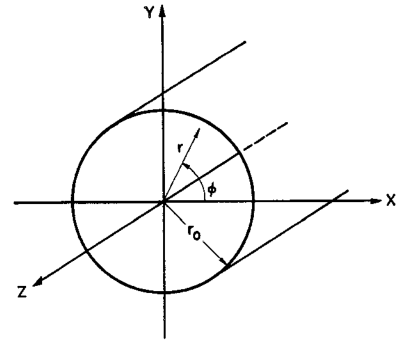

Fig. 2. The cylindrical waveguide.

Substituting (36) or (37) into (27), we find that h must be given by

$$h = -\omega\Omega \pm \sqrt{k^2a^2 - k_c^2a} \tag{38}$$

where

$$k_c^2 = \left(\frac{m\pi}{x_0}\right)^2 + \left(\frac{l\pi}{y_0}\right)^2. \tag{39}$$

Each set of integers m and 1 corresponds to a given mode, which will be designated as the TM_{ml}, or TE_{ml}, mode. The expressions for the electric and magnetic field vectors for the TM modes may be obtained from $\bar{A}$ by means of (28) and (29), and the fields for the TE modes may be obtained from $\bar{F}$ by means of (30) and (31). The results of this calculation are listed in the Appendix.

The Cylindrical Waveguide

For the cylindrical waveguide we consider the geometry shown in Fig. 2. $\bar{A}$ and $\bar{F}$ satisfy (27), which in cylindrical coordinates becomes

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial}{\partial r}\right\} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{1}{a}\frac{\partial^2}{\partial z^2} + \frac{2i\omega\Omega}{a}\frac{\partial}{\partial z} - \frac{\omega^2\Omega^2}{a} + k^2a\right]\left\{\frac{\bar{A}}{\bar{F}}\right\} = 0. \tag{40}$$

The proper solutions for $\bar{A}$ and $\bar{F}$ for this geometry are given by

$$\bar{A} = \hat{z}A = \hat{z}A_0J_m(k_cr)\begin{matrix}\cos\\ \sin\end{matrix}\, m\phi e^{ihz} \tag{41}$$

and

$$\bar{F} = \hat{z}F = \hat{z}F_0J_m(k_cr)\begin{matrix}\cos\\ \sin\end{matrix}\, m\phi e^{ihz} \tag{42}$$

where $J_m(k_cr)$ is the Bessel function of order m. Substituting (41) or (42) into (40), we find that "h" must again satisfy (38), where now k_c is given by

$$k_c = \frac{\rho_{ml}}{r_0} \quad \text{for TM modes} \tag{43}$$

$$k_c = \frac{\rho_{ml}'}{r_0} \quad \text{for TE modes} \tag{44}$$

and where ρ_{ml} denotes the roots of the Bessel function $J_m(\rho)=0$ and ρ_{ml}' denotes the roots of the derivative of the Bessel function $\partial J_m(\rho)/\partial\rho=0$. The subscripts m and l denote, respectively, the order of the Bessel function and the index of the root [5]. The complete expressions for the electric and magnetic fields may be obtained from $\bar{A}$ and (28) and (29) for TM modes, and from $\bar{F}$ and (30) and (31) for TE modes. The results of this calculation are also given in the Appendix.

The Cutoff Behavior

In this section we discuss the propagation characteristics of the waveguide filled with a moving medium. The formulas and conclusions given will apply to both rectangular and cylindrical waveguides (or to a waveguide with any other cross-sectional geometry, if k_c is appropriately defined). k_c will assume the value given in (39) for rectangular waveguides and the value given in (43) or (44) for cylindrical waveguides.

Consider (38) for h. When the velocity of the moving medium is small, so that $n\beta<1$, cutoff will occur if

$$k^2a \leq k_c^2, \tag{45}$$

and hence the cutoff frequency is

$$f_c = \frac{k_c}{2\pi\sqrt{\mu_0\epsilon_0}}\cdot\sqrt{\frac{1-n^2\beta^2}{n^2(1-\beta^2)}}\,. \tag{46}$$

When the frequency is less than f_c, the fields are attenuated along the guide axis, but unlike an ordinary waveguide below cutoff, there is a phase velocity $v_p=1/\Omega$ in the negative z-direction for both solutions. When the frequency is slightly above f_c, there is no attenuation, but the two waves both have phase velocities in the negative z-direction (but they have different phase velocities). Finally, if the frequency is large enough so that

$$k^2a^2 - k_c^2a \geq \omega^2\Omega^2 \tag{47}$$

which can be manipulated to the form

$$f \geq f_+ = \frac{k_c}{2\pi\sqrt{\mu_0\epsilon_0}}\cdot\sqrt{\frac{1-\beta^2}{n^2-\beta^2}}, \tag{48}$$

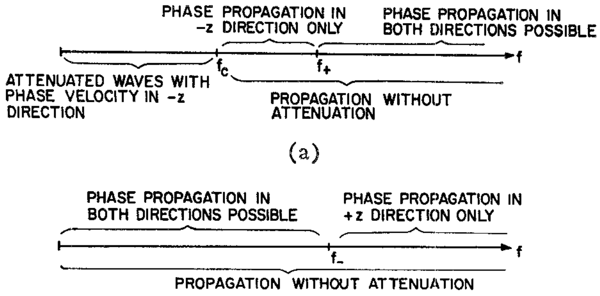

Fig. 3. Frequency ranges for wave propagation in the waveguide, with the medium moving in the $+z$ direction. (a) The low velocity case: $n\beta<1$. (b) The high velocity case: $n\beta>1$.

then waves can propagate in either direction without attenuation, but again with different phase velocities. If $v=0$, then $\beta=0$, and we have

$$f_+ = f_c = \frac{k_c}{2\pi\sqrt{\mu\epsilon}} \tag{49}$$

which is the usual cutoff frequency in the stationary case.

When $n\beta>1$, a will be negative while $-\omega\Omega$ is positive. In this case there will be no cutoff phenomenon at all. At low frequencies, the term contributed by the square root in (38) predominates, so the phase velocities of the two waves are in opposite directions. At higher frequencies $-\omega\Omega$ is always greater than the square root, so both waves have phase velocities in the $+z$ direction. The transition between these two cases occurs at

$$f = f_- = \frac{k_c}{2\pi\sqrt{\mu_0\epsilon_0}} \cdot \sqrt{\frac{1-\beta^2}{n^2-\beta^2}}\,. \tag{50}$$

We note that the relation $n\beta>1$ is the condition for Cerenkov radiation in the medium. A summary of these results is presented in Fig. 3.

There are an infinite number of modes which can exist in the waveguide, but for a given frequency, only a finite number of them can propagate freely, assuming the velocity of the medium in the waveguide is small, so that $n\beta<1$. However, if the velocity of the medium is large enough so that $n\beta>1$, then all modes can propagate freely at any frequency.

Some Relations Between the Waveguide Parameters

For the case $n\beta<1$, several parameters can be expressed in terms of the cutoff frequency:

$$k_c = 2\pi f_c\sqrt{\mu\epsilon a} \tag{51}$$

$$h = -\omega\Omega \pm a\omega\sqrt{\mu\epsilon}\left[1-\left(\frac{f_c}{f}\right)^2\right]^{1/2} \tag{52}$$

$$= \frac{\omega}{(1-n^2\beta^2)c}\left\{(1-n^2)\beta \pm n(1-\beta^2) \cdot \left[1-\left(\frac{f_c}{f}\right)^2\right]^{1/2}\right\} \quad (f\geq f_c) \tag{53}$$

$$h = -\omega\Omega \pm ia\omega_c\sqrt{\mu\epsilon}\left[1-\left(\frac{f}{f_c}\right)^2\right]^{1/2} \tag{54}$$

$$= \frac{1}{(1-n^2\beta^2)c}\left\{\omega(1-n^2)\beta \pm i\omega_c n(1-\beta^2) \cdot \left[1-\left(\frac{f}{f_c}\right)^2\right]^{1/2}\right\} \quad (f\leq f_c) \tag{55}$$

where $\omega_c=2\pi f_c$. For $f>f_c$, the guide phase velocity and guide wavelength are, respectively,

$$v_g = \frac{\omega}{h} = (1-n^2\beta^2)c \Big/ \left\{(1-n^2)\beta \pm n(1-\beta^2) \cdot \left[1-\left(\frac{f_c}{f}\right)^2\right]^{1/2}\right\} \tag{56}$$

$$\lambda_g = \frac{2\pi}{h} = (1-n^2\beta^2)\lambda_0 \Big/ \left\{(1-n^2)\beta \pm n(1-\beta^2) \cdot \left[1-\left(\frac{f_c}{f}\right)^2\right]^{1/2}\right\} \tag{57}$$

where $c=1/\sqrt{\mu_0\epsilon_0}$ and λ_0 is the free space wavelength. The TM_{ml} characteristic wave impedance is

$$Z_{ml}^{TM} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} \quad \text{(rectangular waveguides)} \tag{58}$$

$$= \frac{E_r}{H_\phi} = -\frac{E_\phi}{H_r} \quad \text{(cylindrical waveguides)} \tag{59}$$

$$= \frac{h+\omega\Omega}{\omega\epsilon a}$$

$$= \frac{\pm\sqrt{k^2a^2-k_c^2a}}{\omega\epsilon a} \quad (n\beta<1 \text{ or } n\beta>1) \tag{60}$$

$$= \pm\eta\left[1-\left(\frac{f_c}{f}\right)^2\right]^{1/2} \quad (n\beta<1 \text{ and } f>f_c) \tag{61}$$

$$= \pm i\frac{\omega_c}{\omega}\eta\left[1-\left(\frac{f}{f_c}\right)^2\right]^{1/2} \quad (n\beta<1 \text{ and } f<f_c) \tag{62}$$

where $\eta=\sqrt{\mu/\epsilon}$ is the intrinsic impedance of the medium. Similarly, the TE_{ml} characteristic wave impedance is also given by equations (58) and (59), which for this case yield

$$Z_{ml}^{TE} = \frac{\omega\mu a}{h+\omega\Omega}$$

$$= \pm\frac{\omega\mu a}{\sqrt{k^2a^2-k_c^2a}} \quad (n\beta<1 \text{ or } n\beta>1) \tag{63}$$

$$= \pm\eta\left[1-\left(\frac{f_c}{f}\right)^2\right]^{-1/2} \quad (n\beta<1 \text{ and } f>f_c) \tag{64}$$

$$= \mp i\frac{\omega}{\omega_c}\eta\left[1-\left(\frac{f}{f_c}\right)^2\right]^{-1/2} \quad (n\beta<1 \text{ and } f<f_c). \tag{65}$$

It is interesting to note that the product

$$Z_{ml}^{\mathrm{TM}} Z_{ml}^{\mathrm{TE}} = \eta^2 = \frac{\mu}{\epsilon}$$

at all frequencies and for all velocities of the medium. Z_{ml}^{TM} and Z_{ml}^{TE}, as given in (61) and (62) and in (64) and (65), are of the same form as when the medium in the waveguide is stationary.

The power flow in the rectangular waveguide for TM modes is

$$P = \mathrm{Re}\frac{1}{2}\int_0^{x_0}\int_0^{y_0} \overline{E} \times \overline{H}^* \cdot \overline{dS}$$

$$= \frac{x_0 y_0}{2\epsilon_{0m}\epsilon_{0l}} |A_0|^2 k_c^2 \frac{h + \omega\Omega}{\omega\mu^2\epsilon a^3} \quad (n\beta < 1 \text{ or } n\beta > 1) \qquad (66)$$

$$= \pm \frac{x_0 y_0 k_c^2 |A_0|^2}{2\epsilon_{0m}\epsilon_{0l}} \frac{[1 - (f_c/f)^2]^{1/2}}{a^2\mu\sqrt{\mu\epsilon}} \quad (n\beta < 1) \qquad (67)$$

where ϵ_{0l} is defined as equal to 1 when $l = 0$ and equal to 2 when $l > 0$. For TE modes it is

$$P = \frac{y_0 x_0 k_c^2 |F_0|^2}{2\epsilon_{0m}\epsilon_{0l}} \cdot \frac{h + \omega\Omega}{\omega\mu\epsilon^2 a^3} \quad (n\beta < 1 \text{ or } n\beta > 1) \qquad (68)$$

$$= \pm \frac{x_0 y_0 k_c^2 |F_0|^2}{2\epsilon_{0m}\epsilon_{0l}} \frac{[1 - (f_c/f)^2]^{1/2}}{a^2\epsilon\sqrt{\mu\epsilon}} \quad (n\beta < 1). \qquad (69)$$

In cylindrical waveguides the corresponding expressions are

$$P = \frac{\pi r_0^2 |A_0|^2}{2\epsilon_{0m}} \left[\frac{d}{dr} J_m(k_c r)\Big|_{r=r_0}\right]^2 \frac{h + \omega\Omega}{\omega\mu^2\epsilon a^3} \quad (n\beta < 1 \text{ or } n\beta > 1) \qquad (70)$$

$$= \pm \frac{\pi r_0^2 |A_0|^2}{2\epsilon_{0m}} \left[\frac{d}{dr} J_m(k_c r)\Big|_{r=r_0}\right]^2 \frac{[1 - (f_c/f)^2]^{1/2}}{a^2\mu\sqrt{\mu\epsilon}} \cdot (n\beta < 1) \qquad (71)$$

for TM modes and

$$P = \frac{\pi k_c^2 r_0^2 |F_0|^2}{2\epsilon_{0m}} \left[1 - \frac{m^2}{k_0^2 r_0^2}\right] J_m^2(k_c r_0) \frac{h + \omega\Omega}{\omega\mu\epsilon^2 a^3} \quad (n\beta < 1 \text{ or } n\beta > 1) \qquad (72)$$

$$= \pm \frac{\pi k_c^2 r_0^2 |F_0|^2}{2\epsilon_{0m}} \left[1 - \frac{m^2}{k_c^2 r_0^2}\right] J_m^2(k_c r_0) \frac{[1 - (f_c/f)^2]^{1/2}}{a^2\epsilon\sqrt{\mu\epsilon}} \quad (n\beta < 1) \qquad (73)$$

for TE modes. Although the phase velocities are as shown in Fig. 3, the power flow for the two waves in the guide are, in each case, of the same magnitude and in opposite directions.

When the velocity $\bar{v}$ approaches zero or when the constitutive parameters of the medium equal those of free space Ω will approach zero and a will approach one; all the results obtained then reduce to the familiar ones for the medium at rest.

Conclusions

From the Maxwell-Minkowski equations for the electromagnetic field in a moving medium it has been shown how the electromagnetic fields may be constructed from a pair of vector potential functions $\overline{A}$ and $\overline{F}$, which are derived using a technique similar to that commonly used for stationary media. The solutions for $\overline{A}$ and $\overline{F}$ appropriate to a rectangular and a cylindrical waveguide have been given, as well as the formulas for the fields. These results show the dependence of the fields on the velocity of the medium inside the waveguide.

The propagation constant for the fields in the waveguide has been examined to determine how the motion of the medium affects the cutoff behavior. It was found that the well-known cutoff frequency for a waveguide is modified when the medium inside is moving. For $n\beta < 1$, corresponding to a slowly moving medium, there are two critical frequencies f_c and f_+. For $f < f_c$, the fields are attenuated, as in a conventional waveguide, but also have a phase velocity, unlike a conventional waveguide. For $f_c < f < f_+$, the fields are unattenuated, but all fields have a phase velocity in the $-z$ direction. For $f > f_+$, waves may travel unattenuated in either direction in the waveguide, but with a different phase velocity in each direction. For $n\beta > 1$, corresponding to the case of Cerenkov radiation, there is one critical frequency f_-. For $f < f_-$, waves may travel with a phase velocity in either direction. For $f > f_-$, all solutions have a phase velocity in the $+z$ direction. Also for $n\beta > 1$, there is no cutoff phenomenon in the usual sense. Waves may propagate unattenuated at any frequency.

Finally, some formulas for the waveguide characteristic impedance and for the power flow in a waveguide filled with a moving medium have been given.

Appendix

In this appendix we list the complete expressions for the electric and magnetic fields inside the waveguide.

For the rectangular waveguide shown in Fig. 1, the fields of the TM_{ml} mode may be obtained by substituting $\overline{A}$ given in (36) into (28) and (29). The result is

$$E_x = -\frac{A_0 (h + \omega\Omega)}{\omega\mu\epsilon a^2} \frac{m\pi}{x_0} \cos\frac{m\pi}{x_0} x \sin\frac{l\pi}{y_0} y e^{ihz} \qquad (74)$$

$$E_y = -\frac{A_0(h + \omega\Omega)}{\omega\mu\epsilon a^2} \frac{l\pi}{y_0} \sin\frac{m\pi}{x_0} x \cos\frac{l\pi}{y_0} y e^{ihz} \qquad (75)$$

$$E_z = A_0\left[i\omega + \frac{(h + \omega\Omega)^2}{i\omega\mu\epsilon a^2}\right] \sin\frac{m\pi}{x_0} x \sin\frac{l\pi}{y_0} y e^{ihz} \qquad (76)$$

$$H_x = \frac{A_0}{\mu a} \frac{l\pi}{y_0} \sin\frac{m\pi}{x_0} x \cos\frac{l\pi}{y_0} y e^{ihz} \qquad (77)$$

$$H_y = -\frac{A_0}{\mu a} \frac{m\pi}{x_0} \cos\frac{m\pi}{x_0} x \sin\frac{l\pi}{y_0} y e^{ihz}. \qquad (78)$$

The fields for the TE_{ml} mode are obtained by substituting $\overline{F}$ given in (37) into (30) and (31), with the result:

$$E_x = \frac{F_0}{\epsilon a} \frac{l\pi}{y_0} \cos \frac{m\pi}{x_0} x \sin \frac{l\pi}{y_0} y e^{ihz} \tag{79}$$

$$E_y = -\frac{F_0}{\epsilon a} \frac{m\pi}{x_0} \sin \frac{m\pi}{x_0} x \cos \frac{l\pi}{y_0} y e^{ihz} \tag{80}$$

$$H_x = \frac{F_0(h + \omega\Omega)}{\omega\mu\epsilon a^2} \frac{m\pi}{x_0} \sin \frac{m\pi}{x_0} x \cos \frac{l\pi}{y_0} y e^{ihz} \tag{81}$$

$$H_y = \frac{F_0(h + \omega\Omega)}{\omega\mu\epsilon a^2} \frac{l\pi}{y_0} \cos \frac{m\pi}{x_0} x \sin \frac{l\pi}{y_0} y e^{ihz} \tag{82}$$

$$H_z = F_0\left[i\omega + \frac{(h + \omega\Omega)^2}{i\omega\mu\epsilon a^2}\right] \cos \frac{m\pi}{x_0} x \cos \frac{l\pi}{y_0} y e^{ihz}. \tag{83}$$

For the cylindrical waveguide shown in Fig. 2, the fields of the TM_{ml} mode are obtained from $\overline{A}$ given in (41), and (28) and (29), with the result

$$E_r = -\frac{k_c A_0(h + \omega\Omega)}{\omega\mu\epsilon a^2} \frac{dJ_m(k_c r)}{d(k_c r)} \begin{matrix}\cos \\ \sin\end{matrix}\, m\phi e^{ihz}. \tag{84}$$

$$E_\phi = \pm \frac{A_0 m(h + \omega\Omega)}{\omega\mu\epsilon a^2} \frac{1}{r} J_m(k_c r) \begin{matrix}\sin \\ \cos\end{matrix}\, m\phi e^{ihz} \tag{85}$$

$$E_z = A_0\left[i\omega + \frac{(h + \omega\Omega)^2}{i\omega\mu\epsilon a^2}\right] J_m(k_c r) \begin{matrix}\cos \\ \sin\end{matrix}\, m\phi e^{ihz} \tag{86}$$

$$H_r = \mp \frac{m A_0}{\mu a} \frac{1}{r} J_m(k_c r) \begin{matrix}\sin \\ \cos\end{matrix}\, m\phi e^{ihz} \tag{87}$$

$$H_\phi = -\frac{k_c A_0}{\mu a} \frac{dJ_m(k_c r)}{d(k_c r)} \begin{matrix}\cos \\ \sin\end{matrix}\, m\phi e^{ihz}; \tag{88}$$

and the fields of the TE_{ml} mode are obtained from $\overline{F}$ of (42), (30), and (31):

$$E_r = \pm \frac{F_0 m}{\epsilon a} \frac{1}{r} J_m(k_c r) \begin{matrix}\sin \\ \cos\end{matrix}\, m\phi e^{ihz} \tag{89}$$

$$E_\phi = \frac{k_c F_0}{\epsilon a} \frac{dJ_m(k_c r)}{d(k_c r)} \begin{matrix}\cos \\ \sin\end{matrix}\, m\phi e^{ihz} \tag{90}$$

$$H_r = -\frac{k_c F_0(h + \omega\Omega)}{\omega\mu\epsilon a^2} \frac{dJ_m(k_c r)}{d(k_c r)} \begin{matrix}\cos \\ \sin\end{matrix}\, m\phi e^{ihz} \tag{91}$$

$$H_\phi = \pm \frac{F_0 m(h + \omega\Omega)}{\omega\mu\epsilon a^2} \frac{1}{r} J_m(k_c r) \begin{matrix}\sin \\ \cos\end{matrix}\, m\phi e^{ihz} \tag{92}$$

$$H_z = F_0\left[i\omega + \frac{(h + \omega\Omega)^2}{i\omega\mu\epsilon a^2}\right] J_m(k_c r) \begin{matrix}\cos \\ \sin\end{matrix}\, m\phi e^{ihz}. \tag{93}$$

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